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CIRCLE BUNDLES AND THE KRECK-STOLZ INVARIANT

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ABSTRACT. We present a direct analytic calculation of the s-invariant of Kreck-Stolz for circle bundles, by evaluating the adiabatic limits of η invariants. We believe that this method should have wider applications.

1. INTRODUCTION

Let M be a $4k-1$ dimensional closed spin manifold with vanishing real Pontrjagin classes and a metric of positive scalar curvature. In [KS] Kreck and Stolz introduced a very interesting invariant of such manifold. This so-called s-invariant is an absolute version of a relative invariant introduced by Gromov-Lawson [GL], and plays a critical role in Kreck-Stolz's study of the moduli spaces of positive sectional curvature metrics.

In particular, a calculation of the s-invariant for circle bundles is very crucial for both of the main results in [KS]. This is achieved using cobordism theory in [KS]. In this note we present a direct analytic calculation by evaluating the adiabatic limits of η invariants as well as the characteristic forms appearing in the definition of the s-invariant.

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2. THE s-INVARIANT OF KRECK-STOLZ

Let M be a closed $4k-1$ dimensional spin manifold with vanishing real Pontrjagin classes. Let g be a metric of positive scalar curvature on M. We recall the $\mathbf{Q}$-valued invariant $s(M, g)$ introduced in [KS]. This invariant is related to an integer valued invariant $i(g_1, g_2)$ defined by Gromov and Lawson [GL] for a pair of positive scalar curvature metrics g_1, g_2 on M. More precisely,

$$i(g_1, g_2) = s(M, g_1) - s(M, g_2). \tag{2.1}$$

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These invariants are closely related to the Dirac operator on manifolds with boundary and its index, which explains the integrality or rationality of these invariants.

Remark. The definition of Gromov-Lawson invariant does not require the vanishing of the real Pontryagin classes.

As in [KS], if α, β are two *exact* forms on M, then we define

$$\int_M d^{-1}(\alpha \wedge \beta) = \int_M \hat{\alpha} \wedge \beta, \tag{2.2}$$

where $d\hat{\alpha} = \alpha$. Since β is also exact one verifies easily that the definition does not depend on the choice of $\hat{\alpha}$.

Now if W is a compact manifold with boundary $\partial W = M$, we have the long exact sequence for the de Rham cohomologies:

$$\cdots \to H^*(W, \partial W) \xrightarrow{j} H^*(W) \to H^*(M) \to \cdots. \tag{2.3}$$

Thus if α, β represent relative de Rham classes in $H^*(W, \partial W)$, then $\alpha|_{\partial W} = d\hat{\alpha}$ (and similarly for β). An immediate application of Stokes' Theorem yields

$$\int_M d^{-1}(\alpha \wedge \beta) = \int_W \alpha \wedge \beta - \langle [\alpha] \cup [\beta], [W, \partial W] \rangle. \tag{2.4}$$

Set

$$a_k = \frac{1}{2^{2k+1}(2^{2k-1} - 1)}. \tag{2.5}$$

Denote by $B(M, g)$ (resp. $D(M, g)$) the signature (resp. Dirac) operator on M. We can now define the s-invariant [KS, Definition 2.12].

Definition. The s-invariant $s(M, g)$ is defined as

(2.6)
$$s(M, g) = -\frac{1}{2}\eta(D(M, g)) - a_k \eta(B(M, g)) + \int_M d^{-1}(\hat{A} + a_k L)(p_i(M, g)),$$

where $\eta(D(M, g))$ (resp. $\eta(B(M, g))$) is the η-invariant of $D(M, g)$ (resp. $B(M, g)$) [APS], and $p_i(M, g)$ is the Pontrjagin form obtained via the Chern-Weil theory for the Levi-Civita connection of g.

Remark. The choice of a_k is precisely to cancel out the component in degree $4k$ in $(\hat{A} + a_k L)(p_i(M, g))$, leaving us with a linear combination of products of exact forms.

The usefulness of this invariant comes from the following basic properties of the invariant:

Proposition 2.1 (Kreck-Stolz). *Let M, M' be closed $4k-1$ dimensional spin manifolds with vanishing real Pontrjagin classes and let g, g' be positive scalar curvature metrics on M, M' respectively.*

(a) *If there exists an isometry between (M, g) and (M', g') preserving the spin structures, then $s(M, g) = s(M', g')$.*

(b) *$s(M, g)$ depends only on the connected component of g in the space of metrics of positive scalar curvature on M.*

(c) *If* M *bounds a spin manifold* W *with the metric* g_W *extending* g *and being the product metric near the boundary, then*

$$s(M, g) = \operatorname{ind} D^+(W, g_W) + t(W),$$

where $\operatorname{ind} D^+(W, g_W)$ *denote the index of the Dirac operator on* W *with the Atiyah-Patodi-Singer boundary condition* [APS], *and* $t(W)$ *is a topological invariant defined as*

$$t(W) = -\langle (\hat{A} + a_k L)(j^{-1}p_i(W)), [W, \partial W]\rangle + a_k \operatorname{sign}(W). \tag{2.7}$$

Here j *is the natural map* $j : H^*(W, \partial W) \to H^*(W)$ *from the long exact sequence.*

(d) *The* s*-invariant is additive under connected sum:*

$$s(M\#M', g\#g') = s(M, g) + s(M', g').$$

In the next section, we will give a direct computation of $s(M, g)$ where M is a circle bundle and g is S^1-equivariant.

3. The s-invariant of circle bundles: a computation via adiabatic limit

Let B be a $4k-2$ dimensional closed spin manifold and g^{TB} a metric of positive scalar curvature on B. Let $\pi : N \to B$ be an oriented two dimensional real vector bundle over B and g^N a fiber metric on N with ∇^N a compatible connection. Thus if we denote $R^N = (\nabla^N)^2$ the curvature and $T = \operatorname{Pf}(R^N)$ the Pfaffian, then $\frac{T}{2\pi}$ represents the Euler class e of N.

The connection ∇^N determines a horizontal subbundle $T^H N$ of TN. Let $g^{TN} = g^N \oplus \pi^*(g^{TB})$. Let M be the unit sphere bundle of N with the induced metric g^{TM}. Then M is a circle bundle over B with the holonomy group $U(1)$ acting by isometries and carries an induced spin structure. (This is the spin structure ϕ if we adopt the notation of [KS].)

Since g^{TB} has positive scalar curvature, a standard formula (cf. [KS, (4.4)]) shows that g^{TM} also has positive scalar curvature (this may require shrinking the fiber metric; note that this is compatible with the rescaling in the adiabatic limit defined below). Assume now that all the real Pontrjagin classes of M vanish. The following formula for the s-invariant of M is the key for all the applications in [KS].

Theorem 3.1 (Kreck and Stolz). *The* s*-invariant of* M *is given in terms of the Euler class of* N *and the characteristic classes of* B *as follows.*

$$s(M, \phi, g^{TM}) = -\langle \hat{A}(TB)\frac{1}{2\tanh\frac{e}{2}} + a_k L(TB)\frac{1}{\tanh e}, [B]\rangle + a_k \operatorname{sign}(B_e),$$

where $\operatorname{sign}(B_e)$ *is the signature of the bilinear form*

$$B_e : H^{2k-2}(B) \otimes H^{2k-2}(B) \to R,$$
$$B_e(x \otimes y) = \langle xye, [B]\rangle.$$

This is proved in [KS] by using indirect cobordism techniques.

Since $s(M, \phi, g^{TM})$ is defined in terms of intrinsic analytic invariants, it would be more natural and helpful to provide a direct geometric proof of Theorem 3.1. Using adiabatic limit we present such a proof.

For $\epsilon > 0$, let

$$g_\epsilon = g_\epsilon^{TM} = g^N \oplus \pi^*(\frac{1}{\epsilon} g^{TB}). \tag{3.1}$$

Clearly (M, g_ϵ) still satisfies the requirement in Definition 1.1, so the s-invariant $s(M, g_\epsilon)$ is still defined. Furthermore, (M, g_ϵ) represents a continuous family of metrics of positive scalar curvature. Hence, from (2.1) $s(M, g_\epsilon)$ does not depend on ϵ.

We now take $\epsilon \to 0$. This procedure is referred to as taking the adiabatic limit.

Theorem 3.2. *We have*

$$\lim_{\epsilon\to 0} \frac{1}{2}\eta(D(M, g_\epsilon^{TM})) = -\langle \hat{A}(TB)(\frac{1}{e} - \frac{1}{2\tanh\frac{e}{2}}), [B]\rangle, \tag{3.2}$$

and

$$\lim_{\epsilon\to 0} \eta(B(M, g_\epsilon^{TM})) = \langle L(TB)(\frac{1}{\tanh e} - \frac{1}{e}), [B]\rangle - \operatorname{sign}(B_e). \tag{3.3}$$

Proof. The first result is proved in [Z2, Theorem 2.5], by using the results and methods of Bismut-Cheeger [BC1] and Dai [D1]. The minus sign appears because of the choice of orientation, compare [Z1, Theorem 1]. The other terms disappear because g^{TB} is of positive scalar curvature. Dai [D2] had also independently computed the adiabatic limits of η-invariants of Dirac operators on circle bundles. The novelty of [Z1, Z2] is that Zhang found an application of this result to the Rokhlin type congruences.

For the second formula, let $N_1 = \{u \in N | \|u\|_{g^N} \le 1\}$ be the disc bundle with fibre D over B. Clearly $M = \partial N_1$. It is easy to construct a metric g^{TD} on TD such that for any $\epsilon > 0$, $g_\epsilon^{TN_1} = g^{TD} + \pi^*(\frac{1}{\epsilon} g^{TB})$ is a product near $\partial N_1 = M$ and $g_\epsilon^{TN_1}|_{TM} = g_\epsilon^{TM}$.

Applying the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS] yields, for any $\epsilon > 0$,

$$\operatorname{sign}(N_1) = \int_{N_1} L(P_i(N_1, g_\epsilon^{TN_1})) - \eta(B(M, g_\epsilon^{TM})). \tag{3.4}$$

Or

$$\lim_{\epsilon\to 0} \eta(B(M, g_\epsilon^{TM})) = -\operatorname{sign}(N_1) + \lim_{\epsilon\to 0} \int_{N_1} L(P_i(N_1, g_\epsilon^{TN_1})).$$

But (cf. [BC1])

$$\lim_{\epsilon\to 0} L(P_i(N_1, g_\epsilon^{TN_1})) = L(P_i(B, g^{TB}))L(P_i(D, g^{TD})).$$

Since g^{TD} is a product metric near the boundary, its curvature vanishes near the boundary, and therefore, represents (up to a constant) the Thom class of the vector bundle. Using the Thom isomorphism theorem, a straightforward computation shows

$$\lim_{\epsilon\to 0}\int_{N_1} L(P_i(N_1, g_\epsilon^{TN_1})) = \langle L(TB)(\frac{1}{\tanh e} - \frac{1}{e}), [B]\rangle$$

(compare [Z2, Lemma 3.5]). Also using the Thom isomorphism theorem we have $\operatorname{sign}(N_1) = \operatorname{sign}(B_e)$, proving (3.3). □

Proof of Theorem 3.1. For this purpose it suffices to compute the last term in Definition 1.1, that is

$$\lim_{\epsilon\to 0}\int_M d^{-1}(\hat{A} + a_k L)(p_i(M, g_\epsilon^{TM})).$$

Formula (2.4) gives

$$(3.5)\quad \int_M d^{-1}(\hat{A} + a_k L)(p_i(M, g_\epsilon^{TM})) = \int_{N_1}(\hat{A} + a_k L)(p_i(N_1, g_\epsilon^{TN_1})) - \langle(\hat{A} + a_k L)(p_i(N_1)), [N_1, M]\rangle.$$

Proceeding as above we have

$$(3.6)\quad \lim_{\epsilon\to 0}\int_{N_1}(\hat{A} + a_k L)(p_i(N_1, g_\epsilon^{TN_1})) = \langle\hat{A}(TB)\left(\frac{1}{2\sinh\frac{e}{2}} - \frac{1}{e}\right) + a_k L(TB)\left(\frac{1}{\tanh e} - \frac{1}{e}\right), [B]\rangle.$$

The second term in the right-hand side of (3.5) can be evaluated as in [KS, p. 840], using the bundle splitting $TN_1 = \pi^*(TB) \oplus TD$ and the Thom isomorphism theorem

$$(3.7)\quad \langle j^{-1}(\hat{A} + a_k L)(p_i(N_1)), [N_1, M]\rangle = \langle\hat{A}(TB)\frac{1}{2\sinh\frac{e}{2}} + a_k L(TB)\frac{1}{\tanh e}, [B]\rangle.$$

Combining (3.2), (3.3), and (3.5)–(3.7), we have

$$\lim_{\epsilon\to 0}[\frac{1}{2}\eta(D(M, g_\epsilon^{TM})) + a_k\eta(B(M, g_\epsilon^{TM})) - \int_M d^{-1}(\hat{A} + a_k L)(p_i(M, g_\epsilon^{TM}))] = \langle\hat{A}(TB)\frac{1}{2\tanh\frac{e}{2}} + a_k L(TB)\frac{1}{\tanh e}, [B]\rangle - a_k\operatorname{sign}(B_e).$$

This completes the proof of Theorem 3.1. □

4. Remarks

There is extensive work on the adiabatic limit of eta invariant (and other geometric invariants). In general if M is an oriented manifold that has a fibration structure

$$Y \to M \xrightarrow{\pi} B \tag{4.1}$$

and g_M a submersion metric,

$$g_M = \pi^* g_B + g_Y ,$$

then blowing up the metric in the horizontal direction by a factor x^{-2} gives us a family of metrics g_x,

$$g_x = x^{-2}\pi^* g_B + g_Y .$$

A general formula for $\lim_{x\to 0} \eta(B(M, g_x))$ is given in [D1], which, in fact, comes from a more general formula for Dirac operators (cf. [D1]), namely,

$$\lim_{x\to 0} \eta(A_x) = 2\int_B \mathscr{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + 2\tau , \tag{4.2}$$

where $\tilde{\eta}$ is the the $\tilde{\eta}$-form of Bismut-Cheeger [BC1], R^B is the curvature tensor of g_B and A_B denotes the signature operator on B and A_Y the family of signature operators along Y. The integer τ is a topological invariant computable from the Leray spectral sequence.

More specifically, let (E_r, d_r) $(r \geq 2)$ be the E_r-term of the Leray spectral sequence of the fibration $Y \to M^n \to B$. The orientation gives a natural basis ξ_2 on E_2^n which then induces a basis ξ_r on E_r^n for each $r > 2$. Consider the pairing

$$\langle\ ,\ \rangle_r : E_r^p \otimes E_r^q \longrightarrow \mathbf{R}, \quad \varphi \otimes \psi \longrightarrow (\varphi \cdot d_r \psi , \xi_r). \tag{4.3}$$

If $n = 4k - 1$ (otherwise we set $\tau = 0$) it can be verified that $<, >_r$ is symmetric when restricted to E_r^{2k-1}. Therefore it gives rise to a symmetric matrix whose signature we will denote by τ_r. Define $\tau = \sum_{r\geq 2} \tau_r$.

In the case of circle bundles the terms on the right-hand side of (4.2) can be computed explicitly. For example

$$\tilde{\eta} = 2(\frac{1}{2\tanh\frac{e}{2}} - \frac{1}{e}),$$

and

$$\tau = \mathrm{sign}(B_e).$$

Taking into account of the definition of $\mathscr{L}$ we obtain the same formula as (3.3)

There are other cases where these invariants are quite computable, for example [BC2]. We believe that the method we present above should have wider applications.

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